

Theoretical and numerical studies of a Sobolev-gradient coupled complex boundary method for diffusion coefficient reconstruction

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Abstract We consider the inverse problem of reconstructing spatially varying diffusion coefficients in a general elliptic equation from Cauchy data. Motivated by the domain-integral cost functional arising from the coupled complex boundary method (CCBM), in which the diffusion coefficient does not appear explicitly, we propose a Sobolev-gradient CCBM that incorporates H^1 -type information into both the cost functional and the optimization procedure. The resulting formulation provides an intrinsic stabilization mechanism for the reconstruction process and is combined with Tikhonov regularization over a bounded admissible set of diffusion coefficients. We establish the existence of minimizers, stability with respect to noisy boundary data, and convergence of regularized solutions in the vanishing-noise limit. Numerical experiments with exact and noisy measurements demonstrate that Sobolev-gradient optimization improves reconstruction quality compared with standard L^2 -gradient approaches. Within the scope of the considered numerical examples, the proposed Sobolev-gradient CCBM generally yielded favorable results, reducing oscillatory artifacts while appearing to improve robustness and sensitivity to spatial variations of the diffusion coefficient.

Keywords. Diffusion coefficient reconstruction, coupled complex boundary method, Sobolev gradient.

1 Introduction

Let $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$, be a bounded, simply connected domain with Lipschitz boundary $\Gamma := \partial\Omega$. Let n denote the unit outward normal vector to Γ , and define the normal derivative by $\partial_n := \partial/\partial n$. We consider the inverse problem of recovering a spatially varying diffusion coefficient from Cauchy data for the second-order elliptic equation

$$\mathcal{L}u := -\nabla \cdot (\alpha \nabla u) + b \cdot \nabla u + cu = Q \quad \text{in } \Omega,$$

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where $u: \Omega \rightarrow \mathbb{R}$ denotes the state variable, and the coefficients $\alpha: \Omega \rightarrow \mathbb{R}_+$, $b: \Omega \rightarrow \mathbb{R}^d$, and $c: \Omega \rightarrow \mathbb{R}$ represent the diffusion, advection, and reaction coefficients, respectively, and $Q: \Omega \rightarrow \mathbb{R}$ is a given source term.

The classical inverse conductivity problem (ICP), also known as electrical impedance tomography (EIT), corresponds to the particular case $b = 0$ and $c = 0$ [1, 2, 3, 4]:

$$-\nabla \cdot (\alpha \nabla u) = Q \quad \text{in } \Omega, \quad u = f \quad \text{on } \Gamma. \quad (1.1)$$

We call a pair $(f, g): \Gamma \rightarrow \mathbb{R} \times \mathbb{R}$ the Cauchy data on Γ , where $f = u|_\Gamma$ and $g = \alpha \partial_n u|_\Gamma$ denote the Dirichlet and Neumann data, respectively. Depending on the experimental setting, either the Dirichlet or the Neumann data are prescribed on Γ , while the complementary boundary data are measured. For example, in a Neumann-to-Dirichlet setting, the Neumann data g are prescribed on Γ , and the resulting Dirichlet data f are measured.

Given (f, g) on Γ , the inverse problem we are interested in reads as follows:

Problem 1.1. *Find $\alpha: \Omega \rightarrow \mathbb{R}_+$ such that u solves $\mathcal{L}u = Q$ in Ω with $u = f$ and $\alpha \partial_n u = g$ on Γ .*

Inverse diffusion reconstruction from boundary measurements, as in Problem 1.1, is inherently nonlinear and severely ill-posed [5, 6]. While uniqueness holds given the complete Dirichlet-to-Neumann map [7, 8, 9, 10], recovery from a single Cauchy dataset is generally non-unique and unstable unless further assumptions are enforced. Practical recovery thus relies heavily on regularized optimization strategies.

Widely used formulations include Dirichlet data tracking (TD) and Neumann data tracking (TN), both formulated via least-squares functional fitting, as well as the Kohn–Vogelius (KV) method (see, e.g., [1, 2]) and the coupled complex boundary method (CCBM). Introduced by Cheng et al. [11], the CCBM reformulates the inverse problem via a complex Robin boundary condition, reducing boundary-data matching to minimizing the imaginary component of the state variable. This approach has proven effective across diverse inverse problems [12, 13, 14, 15, 16], including Tikhonov-regularized diffusion coefficient reconstruction [12].

A key characteristic of the CCBM is the form of its cost functional. Whereas TD is defined via boundary integrals, TN and KV both explicitly involve the unknown diffusion coefficient (evaluated on the boundary for TN, and across mixed boundary and interior terms for KV). In contrast, the CCBM functional is expressed as a domain integral of the imaginary part of a complex state, and its integrand contains no explicit dependence on the diffusion coefficient.

Motivated by this observation, we propose a new CCBM-based framework in Section 2 to solve Problem 1.1. Specifically, we augment the CCBM functional with a gradient-weighted term based on the imaginary part of the complex solution. This introduces an H^1 -type stabilization that suppresses noise-induced oscillations. Additionally, using a Sobolev-gradient framework instead of standard L^2 -gradient descent provides extra smoothing during updates.

Theoretically, we analyze the forward map’s continuity and differentiability while proving minimizer existence, stability, and convergence under Tikhonov regularization. For brevity, Section 3 presents only the main convergence result.

Finally, Section 4 presents numerical tests for smooth and piecewise-constant coefficients using exact and noisy data. To our knowledge, this work offers the first numerical benchmark comparing TD, TN, KV, and CCBM for diffusion coefficient reconstruction.

Results confirm that Sobolev-gradient optimization consistently improves reconstruction quality over L^2 -gradient descent in all four methods.

Overall, within the scope of our numerical experiments, the proposed CCBM formulation generally offers a favorable trade-off between reconstruction quality and noise sensitivity. In the examples considered, it consistently reduces oscillatory artifacts while capturing spatial variations in the diffusion coefficient more effectively than the alternative approaches (see Section 5).

2 CCBM setting and forward map analysis

We use the standard spaces, including $L^2(\Omega)$, $L^2(\Gamma)$, and the trace space $H^{1/2}(\Gamma)$ of $H^1(\Omega)$. Unless otherwise specified, the same notation applies to both real- and complex-valued functions, and complex conjugation is omitted when clear from the context. For complex-valued functions, the $H^1(\Omega)$ inner product is defined by $(u, v)_{1,\Omega} := (\nabla u, \nabla v)_{0,\Omega} + (u, v)_{0,\Omega}$, where $(u, v)_{0,\Omega} := \int_{\Omega} u \bar{v} dx$. The associated norm is given by $\|u\|_{1,\Omega}^2 = (u, u)_{1,\Omega}$. Moreover, $\langle \cdot, \cdot \rangle_{\Gamma}$ denotes the duality pairing between $H^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$, coinciding with the $L^2(\Gamma)$ inner product when both arguments belong to $L^2(\Gamma)$. Given $0 < \underline{\alpha} < \bar{\alpha} < \infty$, let

$$\mathcal{A} := \{\alpha \in L^\infty(\Omega) \mid \underline{\alpha} < \alpha < \bar{\alpha} \text{ a.e. in } \Omega\},$$

and denote by $\bar{\mathcal{A}}$ its closure, defined with non-strict bounds $\underline{\alpha} \leq \alpha \leq \bar{\alpha}$. For simplicity, we write $\|\cdot\|_{\infty,\Omega}$ for the $L^\infty(\Omega)$ norm and use a unified notation for scalar- and vector-valued norms.

We identify the diffusion coefficient α via the CCBM reformulation, where Problem 1.1 reduces to requiring the solution's imaginary part to vanish in Ω (see Remark 2.5).

Problem 2.1. *Determine $\alpha \in \bar{\mathcal{A}}$ such that $u_i = 0$ in Ω , where $u = u_r + iu_i: \Omega \rightarrow \mathbb{C}$ solves*

$$\mathcal{L}u = Q \quad \text{in } \Omega, \quad \alpha \partial_n u + iu = g + if \quad \text{on } \Gamma. \quad (2.1)$$

We formulate the variational form of (2.1) under the following assumptions:

Assumption 2.2. (i) $f \in H^{1/2}(\Gamma)$,¹ $g \in H^{-1/2}(\Gamma)$, $Q \in L^2(\Omega)$, $b \in W^{1,\infty}(\Omega)^d$, and $c \in L^\infty(\Omega)$;

(ii) $b \cdot n \geq 0$ a.e. on Γ ;

(iii) $c - \frac{1}{2} \nabla \cdot b \geq \gamma_0 > 0$ a.e. in Ω .

Remark 2.3. *Although the classical case $b = 0, c = 0$ is excluded by Assumption 2.2(iii) and not directly covered by the coercivity results given in Corollary A.2, well-posedness still holds via a Poincaré-type inequality; see [12].*

Under Assumption 2.2, we introduce the weak formulation of the complex boundary value problem (2.1). To this end, define the sesquilinear form $B : H^1(\Omega) \times H^1(\Omega) \rightarrow \mathbb{C}$ and the linear functional $l : H^1(\Omega) \rightarrow \mathbb{C}$ by

$$B(u, v; \alpha) = (\alpha \nabla u, \nabla v)_{0,\Omega} + (b \cdot \nabla u, v)_{0,\Omega} + (cu, v)_{0,\Omega} + i \langle u, v \rangle_{\Gamma},$$

¹While f naturally belongs to the trace space $H^{1/2}(\Gamma)$, treating f and g as independent boundary data enables the weaker assumption $f, g \in H^{-1/2}(\Gamma)$, which guarantees well-posedness of the variational formulation in Problem 2.4 (see Proposition A.1).

$$l(v) = (Q, v)_{0,\Omega} + \langle g, v \rangle_\Gamma + i \langle f, v \rangle_\Gamma,$$

where $f, g : \Gamma \rightarrow \mathbb{R}$.

Problem 2.4. Find $u \in H^1(\Omega)$ such that $B(u, v; \alpha) = l(v)$ for all $v \in H^1(\Omega)$.

Remark 2.5. Problem 2.1 is equivalent to Problem 1.1 for any $\alpha \in \overline{\mathcal{A}}$. Indeed, let $u = u_r + iu_i : \Omega \rightarrow \mathbb{C}$. The real and imaginary parts satisfy:

$$\begin{cases} -\nabla \cdot (\alpha \nabla u_r) + b \cdot \nabla u_r + cu_r = Q & \text{in } \Omega, \\ \alpha \partial_n u_r - u_i = g & \text{on } \Gamma, \end{cases} \quad \begin{cases} -\nabla \cdot (\alpha \nabla u_i) + b \cdot \nabla u_i + cu_i = 0 & \text{in } \Omega, \\ \alpha \partial_n u_i + u_r = f & \text{on } \Gamma. \end{cases}$$

Moreover, “ $u_i \equiv 0$ in Ω ” implies “ (u, α) solves Problem 1.1.” In this case, the Dirichlet data are recovered as $f = u_r$ on Γ , establishing the equivalence.

Under Assumption 2.2, well-posedness of Problem 2.4 follows from the Lax–Milgram Theorem (see, e.g., [17, Theorem 7, p. 368]). A proof sketch and the resulting continuous-dependence estimate (A.1) are provided in Corollary A.2 (Appendix A). Accordingly, the well-posedness result allows us to define the forward map

$$F : \overline{\mathcal{A}} \rightarrow H^1(\Omega), \quad F(\alpha) := u(\alpha), \quad (2.2)$$

where $u(\alpha) = u_r(\alpha) + iu_i(\alpha) \in H^1(\Omega)$ denotes the unique solution of Problem 2.4 corresponding to $\alpha \in \overline{\mathcal{A}}$. When the dependence on α is clear from context, we write u in place of $u(\alpha)$.

We now consider the inverse problem associated with Problem 2.4, which is generally ill-posed. Using Tikhonov regularization with a fixed parameter $\rho > 0$, we minimize the cost functional $J_\rho = J_\rho(\alpha)$ defined by

$$J_\rho(\alpha) := J(\alpha) + \rho R(\alpha) := \frac{1}{2} \left(w_0 \|u_i(\alpha)\|_{0,\Omega}^2 + w_1 \|\nabla u_i(\alpha)\|_{0,\Omega}^2 \right) + \rho \frac{1}{2} \|\alpha\|_{0,\Omega}^2, \quad (2.3)$$

where $w_0 > 0$ and $w_1 \geq 0$ are fixed weighting parameters. The above construction of the cost functional incorporates H^1 -type information through the gradient term $\|\nabla u_i(\alpha)\|_{0,\Omega}^2$ in (2.3). This term is a key feature of the proposed method, extending the conventional CCBM-type misfit functional corresponding to $w_1 = 0$. For sufficiently large ρ , strict convexity of J_ρ holds on the following subset of the admissible set (see Proposition A.7).

Assumption 2.6. The set $\mathcal{K} \subset \mathcal{A}$ is finite-dimensional, closed, and convex.

Under Assumption 2.6, we consider the following minimization problem. Its well-posedness and first-order optimality conditions are established in Theorem 2.8; the proof is provided in Appendix A.

Problem 2.7. Find $\alpha_\rho \in \mathcal{K}$ such that $\alpha_\rho = \arg \min_{\alpha \in \mathcal{K}} J_\rho(\alpha)$.

The regularization parameter $\rho > 0$ plays a crucial role in the well-posedness of the regularized problem. While excessively large values of ρ may over-regularize (i.e., shrink) the reconstructed coefficient, sufficiently large values guarantee coercivity. The following theorem establishes existence, uniqueness, and the corresponding optimality condition.

Theorem 2.8. *Let Assumptions 2.2 and 2.6 hold. We assume that there exists a sufficiently large $\rho_0 > 0$ such that $\rho \geq \rho_0$. Then, Problem 2.7 admits a unique solution $\alpha_\rho \in \mathcal{K}$, that depends continuously on the data. Moreover, α_ρ is characterized by the variational inequality*

$$J'_\rho(\alpha_\rho)(\beta - \alpha_\rho) = (G + \rho\alpha_\rho, \beta - \alpha_\rho)_{0,\Omega} \geq 0, \quad \forall \beta \in \mathcal{K}, \quad (2.4)$$

where the gradient G is given by

$$G = \nabla u_r \cdot \nabla p_i - \nabla u_i \cdot \nabla p_r. \quad (2.5)$$

Here, $u = u_r + iu_i = F(\alpha_\rho)$, and $p = p_r + ip_i \in H^1(\Omega)$ uniquely solves the strong form of the adjoint system, assuming sufficient regularity of u_i :

$$\begin{cases} -\nabla \cdot (\alpha_\rho \nabla p) - \nabla \cdot (bp) + cp = w_0 u_i - w_1 \Delta u_i & \text{in } \Omega, \\ \alpha_\rho \partial_n p + b \cdot np - ip = w_1 \partial_n u_i & \text{on } \Gamma. \end{cases} \quad (2.6)$$

Remark 2.9. *We emphasize that the uniqueness result in Theorem 2.8 holds only for sufficiently large regularization parameters ($\rho \geq \rho_0$). It does not, by itself, guarantee uniqueness for the smaller values of ρ used in Section 4.*

The proof is presented after formulating the adjoint problem for $p \in H^1(\Omega)$. Define the sesquilinear form $B^* : H^1(\Omega) \times H^1(\Omega) \rightarrow \mathbb{C}$ and the linear form $l^* : H^1(\Omega) \rightarrow \mathbb{C}$ by

$$\begin{aligned} B^*(p, v; \alpha_\rho) &= (\alpha_\rho \nabla p, \nabla v)_{0,\Omega} + (p, b \cdot \nabla v)_{0,\Omega} + (cp, v)_{0,\Omega} - i\langle p, v \rangle_\Gamma, \\ l^*(v) &= w_0(u_i, v)_{0,\Omega} + w_1(\nabla u_i, \nabla v)_{0,\Omega}. \end{aligned}$$

Then, the weak formulation of the adjoint system (2.6), whose well-posedness follows from the Lax–Milgram Theorem, can be stated as follows:

Problem 2.10. *Find $p \in H^1(\Omega)$ such that $B^*(p, v; \alpha) = l^*(v)$ for all $v \in H^1(\Omega)$.*

Proof of Theorem 2.8. Since J_ρ is strictly convex on $\mathcal{K}$, standard results on convex minimization (see, e.g., [18, Theorem 5.3.19]) guarantee a unique minimizer $\alpha_\rho \in \mathcal{K}$. We now characterize this solution via the first-order variational inequality:²

$$J'_\rho(\alpha_\rho)(\beta - \alpha_\rho) = J'(\alpha_\rho)(\beta - \alpha_\rho) + \rho R'(\alpha_\rho)(\beta - \alpha_\rho) \geq 0, \quad \forall \beta \in \mathcal{K}. \quad (2.7)$$

Here, the Gâteaux derivative components in an arbitrary direction β are given by

$$J'(\alpha)\beta = w_0(u_i(\alpha), u'_i)_{0,\Omega} + w_1(\nabla u_i(\alpha), \nabla u'_i)_{0,\Omega}, \quad R'(\alpha)\beta = (\alpha, \beta)_{0,\Omega},$$

where $u'_i = \Im\{F'(\alpha)\beta\}$ solves the linearized state problem (Proposition A.4).

Let $p \in H^1(\Omega)$ solve Problem 2.10. The linearized state problem and its adjoint can be written concisely as

$$B(u', p; \alpha_\rho) = -((\beta - \alpha_\rho) \nabla u, \nabla p)_{0,\Omega}, \quad B(u', p; \alpha_\rho) = \overline{B^*(p, u'; \alpha_\rho)} = \overline{l^*(u')}.$$

Using integration by parts, the terms involving b cancel exactly. Taking the imaginary part yields

$$\begin{aligned} J'(\alpha_\rho)(\beta - \alpha_\rho) &= \Im\{w_0(u_i, u')_{0,\Omega} + w_1(\nabla u_i, \nabla u')_{0,\Omega}\} \\ &= \int_\Omega (\beta - \alpha_\rho)(\nabla u_r \cdot \nabla p_i - \nabla u_i \cdot \nabla p_r) dx. \end{aligned}$$

Combining this with (2.7) yields (2.4), completing the proof. $\square$

²Note, however, that because the regularization parameter ρ cannot be chosen arbitrarily large in practice, global convexity of the objective functional cannot be expected in general.

Remark 2.11. *The difference $\beta - \alpha_\rho$ belongs to $L^\infty(\Omega)$.*

3 Convergence analysis under noisy measurements

We now consider the discrete regularized inverse problem (Problem 3.2) with noisy data to recover a stable approximation of the diffusion coefficient in $\mathcal{K}$. To this end, we introduce the following assumption on $\mathcal{K}$:

Assumption 3.1. *There exists a unique $\alpha^\dagger \in \mathcal{K}$ such that $u^\dagger = F(\alpha^\dagger)$ satisfies $u_i^\dagger = 0$ in Ω , i.e., $\alpha^\dagger$ is the exact coefficient producing the measured Dirichlet data f on Γ .*

Let $f^\delta = f + \eta$ with $\|\eta\|_{0,\Gamma} \leq \delta$, where $\delta > 0$ denotes the noise level. Replacing f with f^δ in Problem 2.4, we obtain the perturbed forward problem: find $u^\delta \in H^1(\Omega)$ such that:

$$B(u^\delta, v; \alpha) = (Q, v)_{0,\Omega} + \langle g, v \rangle_\Gamma + i \langle f^\delta, v \rangle_\Gamma, \quad \forall v \in H^1(\Omega).$$

By Corollary A.2, there exists a constant $C > 0$, independent of α and δ , such that $\|u^\delta - u\|_{1,\Omega} \leq C\delta$. The corresponding Tikhonov functional is defined as

$$J_\rho^\delta(\alpha) := \frac{1}{2}(w_0 \|u_i^\delta(\alpha)\|_{0,\Omega}^2 + w_1 \|\nabla u_i^\delta(\alpha)\|_{0,\Omega}^2) + \frac{\rho}{2} \|\alpha\|_{0,\Omega}^2, \quad \alpha \in \mathcal{K},$$

where $u^\delta(\alpha) = u_r^\delta(\alpha) + iu_i^\delta(\alpha)$ denotes the solution of the perturbed forward problem associated with α .

Problem 3.2. *Find $\alpha_\rho^\delta \in \mathcal{K}$ such that $J_\rho^\delta(\alpha_\rho^\delta) = \min_{\alpha \in \mathcal{K}} J_\rho^\delta(\alpha)$.*

Following the framework in [12, 19], Problem 3.2 is well-posed (minimizers exist and depend continuously on data) and converges as noise vanishes. For brevity, we state only the convergence theorem below.

Theorem 3.3. *Let Assumptions 2.2, 2.6, and 3.1 hold. Let $\delta_n \rightarrow 0$ and $\rho_n = \rho(\delta_n) \rightarrow 0$, as $n \rightarrow \infty$. For each n , let $\alpha_{\rho_n}^{\delta_n} \in \mathcal{K}$ be a minimizer of $J_{\rho_n}^{\delta_n}$. Then, $\alpha_{\rho_n}^{\delta_n} \rightarrow \alpha^\dagger$ strongly in $L^2(\Omega)$ and $u^{\delta_n}(\alpha_{\rho_n}^{\delta_n}) \rightharpoonup u^\dagger$ weakly in $H^1(\Omega)$ as $n \rightarrow \infty$.*

Remark 3.4. *We emphasize that the convergence result in Theorem 3.3 is conditional on the uniqueness of the exact coefficient in $\mathcal{K}$, as required by Assumption 3.1.*

Proof of Theorem 3.3. We consider a sequence of vanishing noise levels $\delta_n \rightarrow 0$ as $n \rightarrow \infty$, with corresponding noisy measurement data f^{δ_n} and regularization parameters $\rho_n = \rho(\delta_n) \rightarrow 0$. For each $n \in \mathbb{N}$, denote by $\alpha^n := \alpha_{\rho_n}^{\delta_n} \in \mathcal{K}$ the minimizer of $J_{\rho_n}^{\delta_n}$ corresponding to f^{δ_n} , and set

$$u^n := F^{\delta_n}(\alpha^n) \in H^1(\Omega), \quad u_i^n := \Im\{u^n\}.$$

Since $\mathcal{K} \subset L^2(\Omega)$ is finite-dimensional and compact, there exist a subsequence (not relabeled) and a limit $\bar{\alpha} \in \mathcal{K}$ such that

$$\alpha^n \rightarrow \bar{\alpha} \quad \text{strongly in } L^2(\Omega). \quad (3.1)$$

By the equivalence of norms on the finite-dimensional space $\mathcal{K}$, this also implies $\alpha^n \rightarrow \bar{\alpha}$ strongly in $L^\infty(\Omega)$.

By Corollary A.2, the sequence $\{u^n\}$ is uniformly bounded in $H^1(\Omega)$. Hence, there exist $u^\infty \in H^1(\Omega)$ and a subsequence such that

$$u^n \rightharpoonup u^\infty \quad \text{weakly in } H^1(\Omega), \quad u^n \rightarrow u^\infty \quad \text{strongly in } L^2(\Omega). \quad (3.2)$$

For each fixed $\alpha \in \mathcal{K}$, the continuity of the forward map with respect to data (Corollary A.2 and Proposition A.3) ensures that $F^{\delta_n}(\alpha) \rightarrow F(\alpha)$ in $H^1(\Omega)$ as $\delta_n \rightarrow 0$. Combining this with (3.1) and (3.2), passing to the limit in the variational formulation of the state equation yields $u^\infty = F(\bar{\alpha})$.

Next, since α^n minimizes J_ρ^n over $\mathcal{K}$, we have

$$J_\rho^n(\alpha^n) \leq J_\rho^n(\alpha^\dagger), \quad \forall n \in \mathbb{N}, \quad (3.3)$$

where $\alpha^\dagger \in \mathcal{K}$ is the exact solution (Assumption 3.1). Since $\alpha^\dagger$ is exact, $F(\alpha^\dagger)$ is real-valued ($\Im\{F(\alpha^\dagger)\} = 0$). By continuity of the forward map with respect to the data and $\rho_n \rightarrow 0$, it follows that

$$J_\rho^n(\alpha^\dagger) = \frac{1}{2} \left(w_0 \|\Im\{F^{\delta_n}(\alpha^\dagger)\}\|_{0,\Omega}^2 + w_1 \|\nabla \Im\{F^{\delta_n}(\alpha^\dagger)\}\|_{0,\Omega}^2 \right) + \frac{\rho_n}{2} \|\alpha^\dagger\|_{0,\Omega}^2 \longrightarrow 0. \quad (3.4)$$

By weak lower semicontinuity of norms, together with (3.3) and (3.4), it follows that

$$0 \leq \frac{1}{2} \left(w_0 \|\Im\{F(\bar{\alpha})\}\|_{0,\Omega}^2 + w_1 \|\nabla \Im\{F(\bar{\alpha})\}\|_{0,\Omega}^2 \right) \leq \liminf_{n \rightarrow \infty} J_\rho^n(\alpha^n) \leq \lim_{n \rightarrow \infty} J_\rho^n(\alpha^\dagger) = 0.$$

Hence, $\Im\{F(\bar{\alpha})\} = 0$ in Ω , so $(\bar{\alpha}, F(\bar{\alpha}))$ solves the exact inverse problem. By uniqueness (Assumption 3.1), we conclude $\bar{\alpha} = \alpha^\dagger$ and $u^\infty = F(\alpha^\dagger) = u^\dagger$.

Since the limit $(\alpha^\dagger, u^\dagger)$ is unique, a standard subsequence argument establishes that the entire sequences converge:

$$\alpha_{\rho_n}^{\delta_n} \rightarrow \alpha^\dagger \quad \text{strongly in } L^2(\Omega) \text{ and } L^\infty(\Omega),$$

and

$$F^{\delta_n}(\alpha_{\rho_n}^{\delta_n}) \rightharpoonup u^\dagger \quad \text{weakly in } H^1(\Omega).$$

This completes the proof. $\square$

4 Numerical Approximation

We solve Problem 1.1 by minimizing the regularized cost functional using an H^1 -gradient descent method [20]. Specifically, we compute the H^1 -Riesz representative of the L^2 -gradient G given in (2.5). For a smoothing parameter $\mu > 0$, the corresponding H^1 -gradient $\tilde{G} = \tilde{G}(\alpha) \in H^1(\Omega)$ uniquely satisfies

$$\mu \int_\Omega \nabla \tilde{G} \cdot \nabla \varphi \, dx + \int_\Omega \tilde{G} \varphi \, dx = \int_\Omega G \varphi \, dx, \quad \forall \varphi \in H^1(\Omega), \quad (4.1)$$

which corresponds to the boundary value problem $-\mu \Delta \tilde{G} + \tilde{G} = G$ in Ω with homogeneous Neumann boundary conditions $\partial_n \tilde{G} = 0$ on $\partial\Omega$. If $\mu = 0$, then $\tilde{G} = G$ a.e. in Ω .

Starting from an initial guess $\alpha^{[0]}$, the unprojected gradient step for the regularized functional reads

$$\alpha^{[k+1]} = \alpha^{[k]} - t^{[k]} \left(\tilde{G}(\alpha^{[k]}) + \rho \alpha^{[k]} \right), \quad k = 0, 1, 2, \dots, \quad (4.2)$$

where $t^{[k]} > 0$ is the step size. To enforce admissibility ($\alpha^{[k]} \in \mathcal{K}$), we project the iterates onto $\mathcal{K}$. The complete projected H^1 -gradient procedure, implemented in FREEFEM++ [21], is outlined in Algorithm 1.

Algorithm 1 Projected H^1 -gradient descent algorithm for parameter identification

Require: objective function J , initial guess $\alpha^{[0]} \in \mathcal{K}$, step size parameter $\tau > 0$, maximum iterations $k_{\max}$

Ensure: approximate minimizer α^* (estimate of the exact diffusion coefficient $\alpha^\dagger$)

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1: Set  $k = 0$ 
2: while  $k \leq k_{\max}$  do
3:   Solve the state equation for  $\alpha^{[k]}$  to obtain  $u(\alpha^{[k]})$ 
4:   Solve the adjoint equation (if applicable) to obtain  $p(\alpha^{[k]})$ 
5:   Compute the  $H^1$ -Sobolev gradient  $\tilde{G}(\alpha^{[k]})$  using  $u(\alpha^{[k]})$  and  $p(\alpha^{[k]})$  via (4.1)
6:   Set initial step size  $t^{[k]} = \tau J^{[k]} / \|\tilde{G}(\alpha^{[k]})\|_{0,\Omega}$ 
7:   while  $J(\alpha^{[k]} - t^{[k]} \tilde{G}(\alpha^{[k]})) > J(\alpha^{[k]})$  do ▷ Backtracking line search [22]
8:     Set  $t^{[k]} \leftarrow t^{[k]}/2$ 
9:   end while
10:  Perform the update:  $\tilde{\alpha}^{[k+1]} = \Pi_{\mathcal{K}}(\alpha^{[k]} - t^{[k]} \tilde{G}(\alpha^{[k]}) - t^{[k]} \rho \alpha^{[k]})$ 
11:  Set  $\alpha^{[k+1]} = \Pi_{\text{pc}}(\tilde{\alpha}^{[k+1]})$  if  $\alpha^*$  is piecewise constant, else  $\alpha^{[k+1]} = \tilde{\alpha}^{[k+1]}$  ▷ See Subsection 4.3
12:   $k \leftarrow k + 1$ 
13: end while
14: return  $\alpha^{[k]}$  as  $\alpha^*$ 

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In line 6 of Algorithm 1, the step-size parameter is set to $\tau = 0.5$ in all experiments unless otherwise stated.

The projection operator $\Pi_{\mathcal{K}}: C(\bar{\Omega}) \rightarrow C(\bar{\Omega})$ in Algorithm 1 is defined pointwise at each mesh node P by

$$(\Pi_{\mathcal{K}}\alpha)(P) = \max \{ \min \{ \alpha(P), \bar{\alpha} \}, \underline{\alpha} \},$$

for every nodal point P in the triangulation of $\bar{\Omega}$. The algorithm terminates after a fixed maximum number of iterations, $k_{\max}$. Discrepancy principles and tolerance-based stopping rules are excluded so that early stopping does not act as an implicit regularizer.

Remark 4.1. While w_1 alters the objective functional by scaling the gradient term ∇u_i in the discrepancy functional (2.3), the free parameter μ in (4.1) leaves the objective unchanged and instead defines the inner product metric for the Sobolev derivative. Specifically, increasing μ filters high-frequency noise from the L^2 -gradient G , yielding a smoother search direction $\tilde{G}$ and enhancing numerical stability (Figures 2 and 4).

Remark 4.2. At the discrete level, the H^1 -gradient $\tilde{G}$ is the Riesz representative of the derivative of the cost functional in the space of P_1 finite elements, equipped with the discrete H^1 -inner product.

Before presenting the numerical results, we describe how the synthetic Cauchy data are generated and how noise is added. The forward problem is solved with $\alpha = \alpha^*$ on a fine mesh using P_2 finite elements to generate a high-resolution reference solution u^*

under the prescribed Neumann boundary condition g ; the corresponding exact Dirichlet trace is $f^* = u^*|_I$. To avoid inverse crimes [23, p. 154], the inverse problem is solved on a coarse mesh, and all variational formulations in the inversion process are discretized using P_1 finite elements.

To simulate measurement noise in realistic settings, we perturb only the Dirichlet trace f^* , while keeping the Neumann datum g noise-free to represent a known, exact boundary excitation. Let $\sigma := \|u^*\|_{L^\infty(\Omega)}$. At each vertex x_j ($j = 1, \dots, N_h$) of the fine mesh, we draw an independent random variable $\zeta_j \sim \mathcal{N}(0, \sigma^2)$ and set

$$u^\delta(x_j) = (1 + \delta \zeta_j) u^*(x_j), \quad j = 1, \dots, N_h,$$

where $\delta \geq 0$ denotes the relative noise level and $N_h \in \mathbb{N}$ is the total number of nodal points. The noisy Dirichlet datum used for inversion is $f^\delta := u^\delta|_I$, obtained by interpolating u^δ onto the coarse mesh. Unless noted otherwise, all numerical figures correspond to a single realization of the noise sequence (ζ_j) , rather than an average over multiple realizations.

For each method $M \in \{\text{CCBM}, \text{KV}, \text{TD}, \text{TN}\}$, the gradient norm reported in the figures throughout the following subsections is defined as

$$\|\nabla J_M(\alpha)\|_{0,\Omega} = \|G_M(\alpha)\|_{0,\Omega} = \left(\int_{\Omega} |G_M(\alpha)(x)|^2 dx \right)^{1/2},$$

where $G_M = \nabla J_M$ denotes the corresponding gradient defined in Appendix B.

4.1 Reconstruction of smooth diffusion

Let $C(a, r) := \{x = (x_1, x_2)^\top \in \mathbb{R}^2; |x - a| < r\}$ for $a \in \mathbb{R}^2$ and $r > 0$. We consider $\Omega = C(0, 1)$ with $\alpha^* = 1 + 0.5x_1x_2$, $\alpha^{[0]} = 1$, $b = 0$, $c = 1$, and $Q = 1$. Unless otherwise stated, we primarily set $w_1 = 0$ to facilitate comparison with traditional methods. The effect of nonzero w_1 is examined separately to highlight the improvement provided by the additional gradient information; see subsection 4.2.

Figure 1 presents initial tests comparing the four reconstruction methods under noise-free ($\delta = 0$) and noisy ($\delta = 10^{-3}$) observations without explicit regularization and stabilization ($\mu = \rho = 0$). To maintain stability, a small step size $\tau = 0.01$ is used.

Without noise, all methods produce reasonable reconstructions, though CCBM yields the smallest state and coefficient errors and the fastest gradient decay. With noisy data, CCBM's advantage becomes more pronounced, achieving the lowest reconstruction errors while competing methods stagnate or diverge. These results demonstrate that CCBM remains robust without regularization.

For all subsequent experiments, we fix $\tau = 0.1$.

Figure 2 examines the role of μ . Moderate values of μ produce accurate, oscillation-free reconstructions with monotonic convergence for CCBM, KV, and TD, though TN remains sensitive to small values of μ . The effect of the Tikhonov regularization parameter ρ is reported in Figure C.1 in Appendix C. While smaller values of ρ ($\delta = 0.0003$) tend to induce oscillatory reconstructions, particularly without H^1 -gradient smoothing, CCBM remains substantially more stable than the other formulations. Moreover, when H^1 -gradient smoothing is incorporated, all methods become more stable, with CCBM generally exhibiting favorable robustness and reconstruction quality among the examples considered.

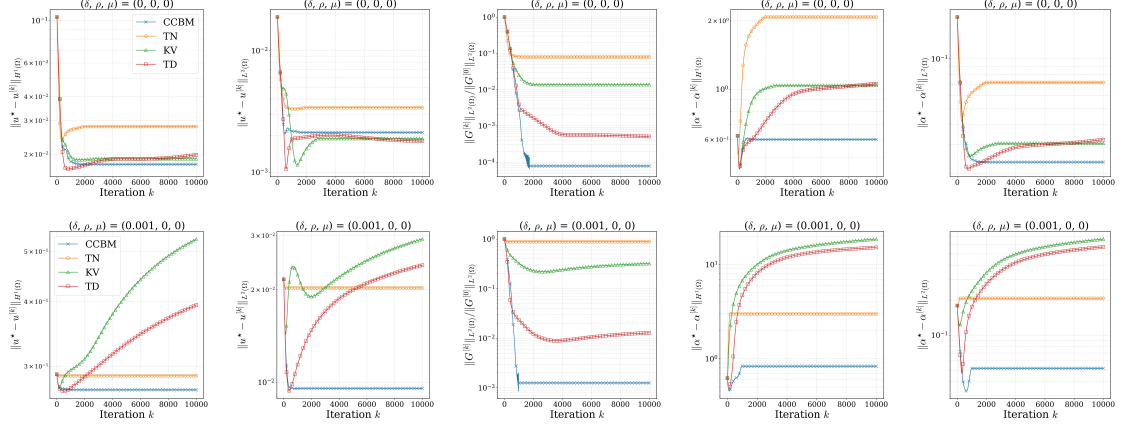


Figure 1: Convergence histories for CCBM, KV, TD, and TN for $\delta = 0$ (top) and $\delta = 10^{-3}$ (bottom). Left to right: relative state errors (H^1, L^2), normalized gradient norm, and relative coefficient errors (H^1, L^2) vs. iteration.

We next consider the mildly oscillatory diffusion $\alpha^\star(x) = 1 + 0.25 \sin(\pi x_1) \sin(\pi x_2)$ on $\Omega = C(0, 2)$ with boundary input $g(x) = 2 + \sin(\pi x_1) \sin(\pi x_2)$, noise level $\delta = 0.001$ and gradient smoothing $\mu = 1.0$, and $\rho = 10^{-8}$.

Figure 3 shows that, among the examples considered, CCBM provides accurate reconstruction under oscillatory boundary inputs, while KV and TN exhibit stronger boundary artifacts and TD produces mild distortions near the boundary.

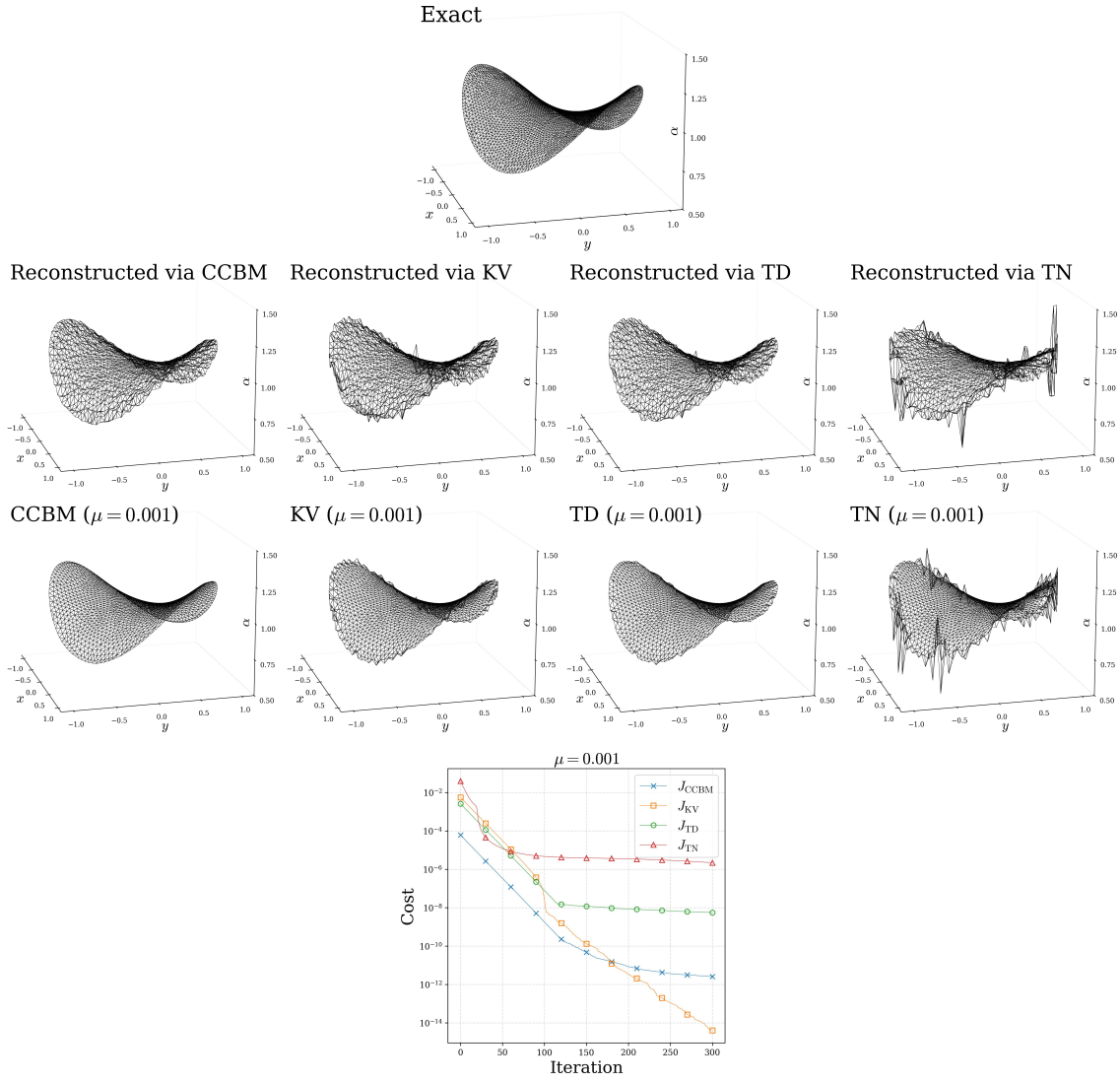


Figure 2: Reconstructions of α^* (top) via CCBM, KV, TD, and TN without H^1 smoothing (second row) and with H^1 smoothing ($\mu = 0.001$, third row). Bottom: corresponding cost history.

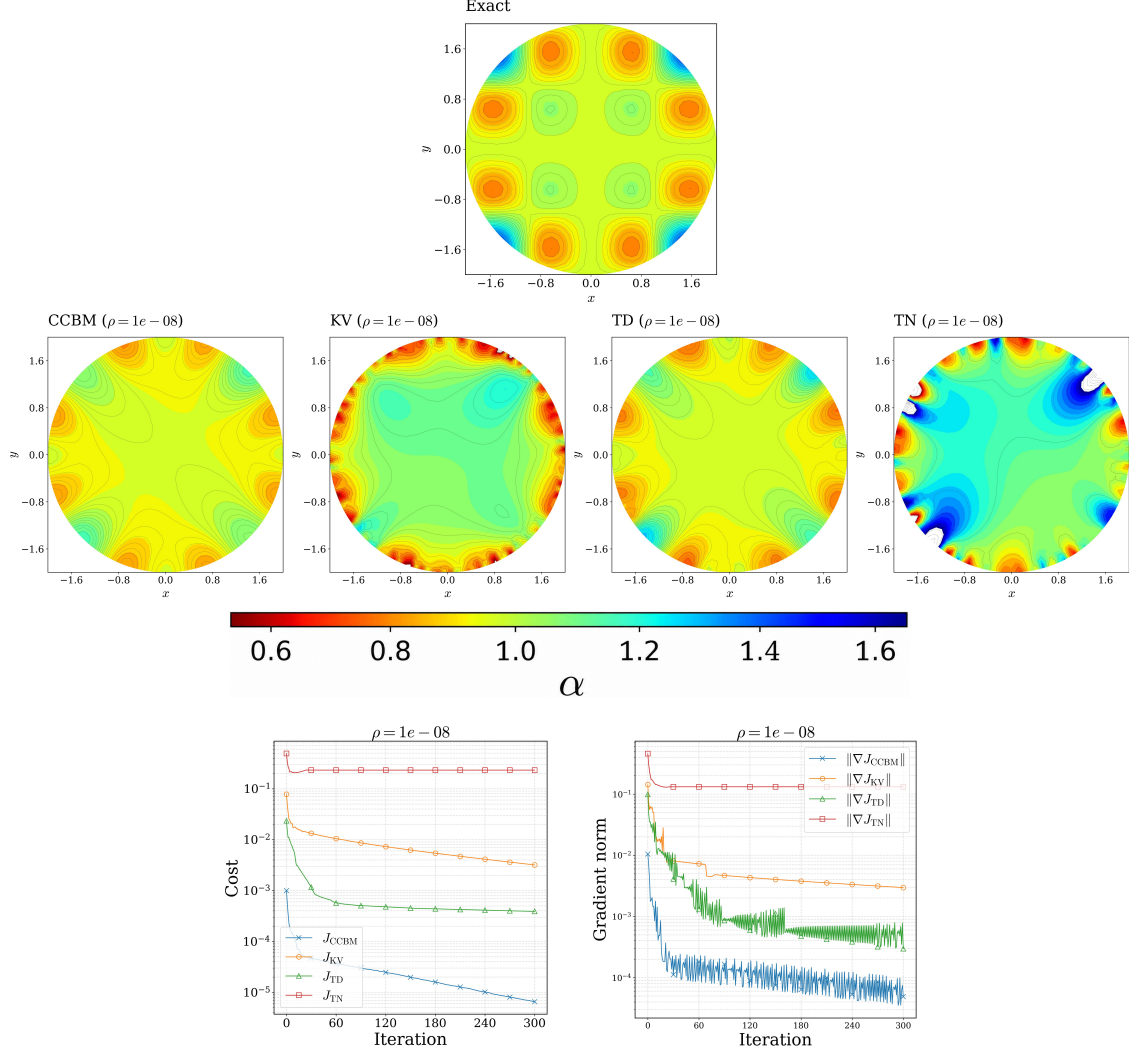


Figure 3: Reconstructions of $\alpha^*(x) = 1 + 0.25 \sin(\pi x_1) \sin(\pi x_2)$ ($\delta = 0.001$, $g(x) = 2 + \sin(\pi x_1) \sin(\pi x_2)$). Top: exact profile. Middle row: CCBM, KV, TD, and TN (left to right) reconstructions, with the shared colorbar shown below. Bottom row: cost (left) and gradient norm (right) histories.

4.2 Effect of incorporating the H^1 term in the cost functional

We investigate the effect of the gradient weight w_1 in the modified CCBM functional (2.3) with $w_0 = 1$. Let $\Omega = C(0, 2)$ and consider the exact diffusion coefficient $\alpha_1^*(x, y) = 1 + 0.5xy$ or $\alpha_2^*(x, y) = 1 + \sin(\pi x/8) \sin(\pi y/8)$. We set $b = 0$, $Q = 0$, $c = 1$, use $g_1 = 1$ or $g_2 = \sin(\pi x) \sin(\pi y)$, consider $\delta = 0, 0.001$, and take $\mu = 0.1$, $\rho = 0.001$, and $\alpha^{[0]} \equiv 1$.

Figure 4 shows the effect of w_1 . In most cases, intermediate parameter values minimize both the absolute L^2 and H^1 errors, $\|\alpha^* - \alpha^{[k_{\max}]}\|_{0,\Omega}$ and $\|\alpha^* - \alpha^{[k_{\max}]}\|_{1,\Omega}$, respectively. In contrast, values that are too small provide insufficient sensitivity to the spatial variation of u , while values that are too large place excessive emphasis on its gradient discrepancy. This behavior is robust across diffusion profiles, boundary conditions, and noise levels, with the optimal range remaining problem-dependent. For noisy data, the H^1 term improves stability by suppressing high-frequency artifacts.

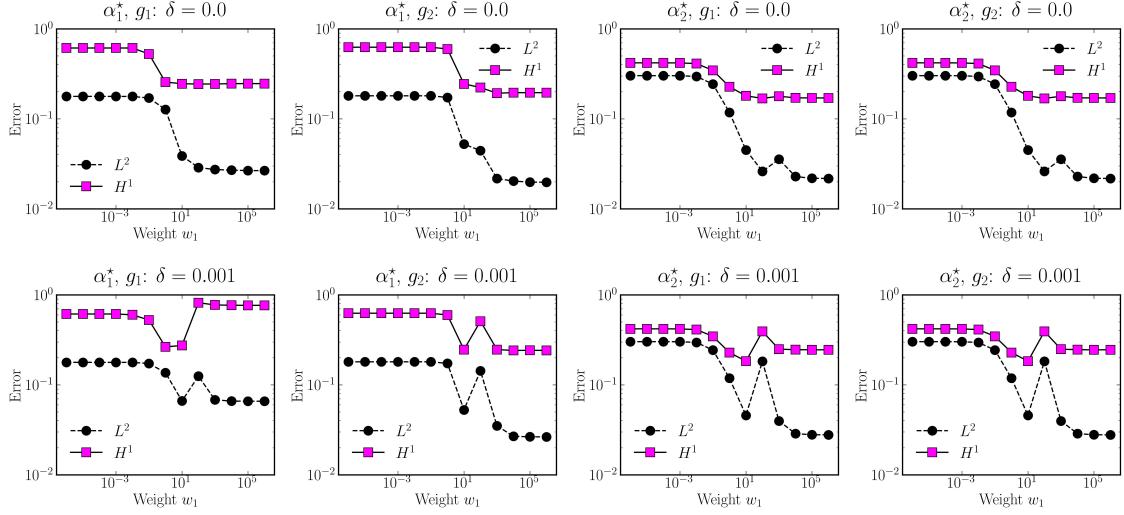


Figure 4: Effect of the weight w_1 for $(\alpha^*, g) = (\alpha_1^*, g_1)$, (α_1^*, g_2) , (α_2^*, g_1) , (α_2^*, g_2) (left to right). Top row: $\delta = 0$; bottom row: $\delta = 0.001$.

4.3 Piecewise-constant diffusion coefficient estimation on a prescribed partition

In this experiment, the partition of the domain into subregions is prescribed a priori, and only the constant coefficient value on each subregion is estimated. Thus, the experiment concerns piecewise-constant coefficient estimation on a known partition and does not involve the recovery of unknown interfaces.

The projection Π_{pc} in Line 11 of Algorithm 1 uses a *pick-a-point* strategy: $\tilde{\alpha}^{[k+1]}$ is sampled at a fixed interior point per subregion and assigned uniformly over it (Sobolev gradient in P_1 , coefficient in P_0). This effectively enforces piecewise-constant structure for simple geometries when combined with Sobolev smoothing.

We let $b = 0$, $c = 1$, $Q(x_1, x_2) = x_1 + x_2 + 2$, $g(x_1, x_2) = \exp(\sin(\pi x_1) \sin(\pi x_2))$, $\alpha^{[0]} = 2$, and $\Omega = (-1, 1)^2$. For $R = 0.75$, the exact coefficient is defined on subregions Ω_C , Ω_L , and Ω_R as

$$\alpha^*(x_1, x_2) = \begin{cases} 1.50, & (x_1, x_2) \in \Omega_C := \{x_1^2 + x_2^2 \leq R^2\}, \\ 0.75, & (x_1, x_2) \in \Omega_L := \{x_1 < 0, x_1^2 + x_2^2 > R^2\}, \\ 0.50, & (x_1, x_2) \in \Omega_R := \{x_1 \geq 0, x_1^2 + x_2^2 > R^2\}. \end{cases}$$

This configuration is shown in Figure 5(a). The pick-a-point restriction uses $(0, 0)$, $(-0.95, 0)$, and $(0.95, 0)$ for Ω_C , Ω_L , and Ω_R , respectively.

Figure 5(b)–(d) shows the reconstruction performance for $\delta = 0.01$. Although only one case is shown, we observe that as δ increases, the errors grow and become more pronounced in subregions farther from the boundary. Additional numerical experiments, including the two-subregion case presented in Appendix C (Figure C.2), indicate that CCBM generally exhibits favorable stability and noise robustness among the examples considered. While TD is slightly more accurate at low noise levels, CCBM performs better as δ increases (Figure 5(e)). Here, the *average error* is given by

$$\text{Avg. error} = \frac{1}{3} \sum_{i=L,C,R} |\alpha_{\Omega_i} - \alpha_{\Omega_i}^*|$$

at each fixed noise level δ (single realization per δ), and is distinct from an average over noise realizations or over coefficient values.

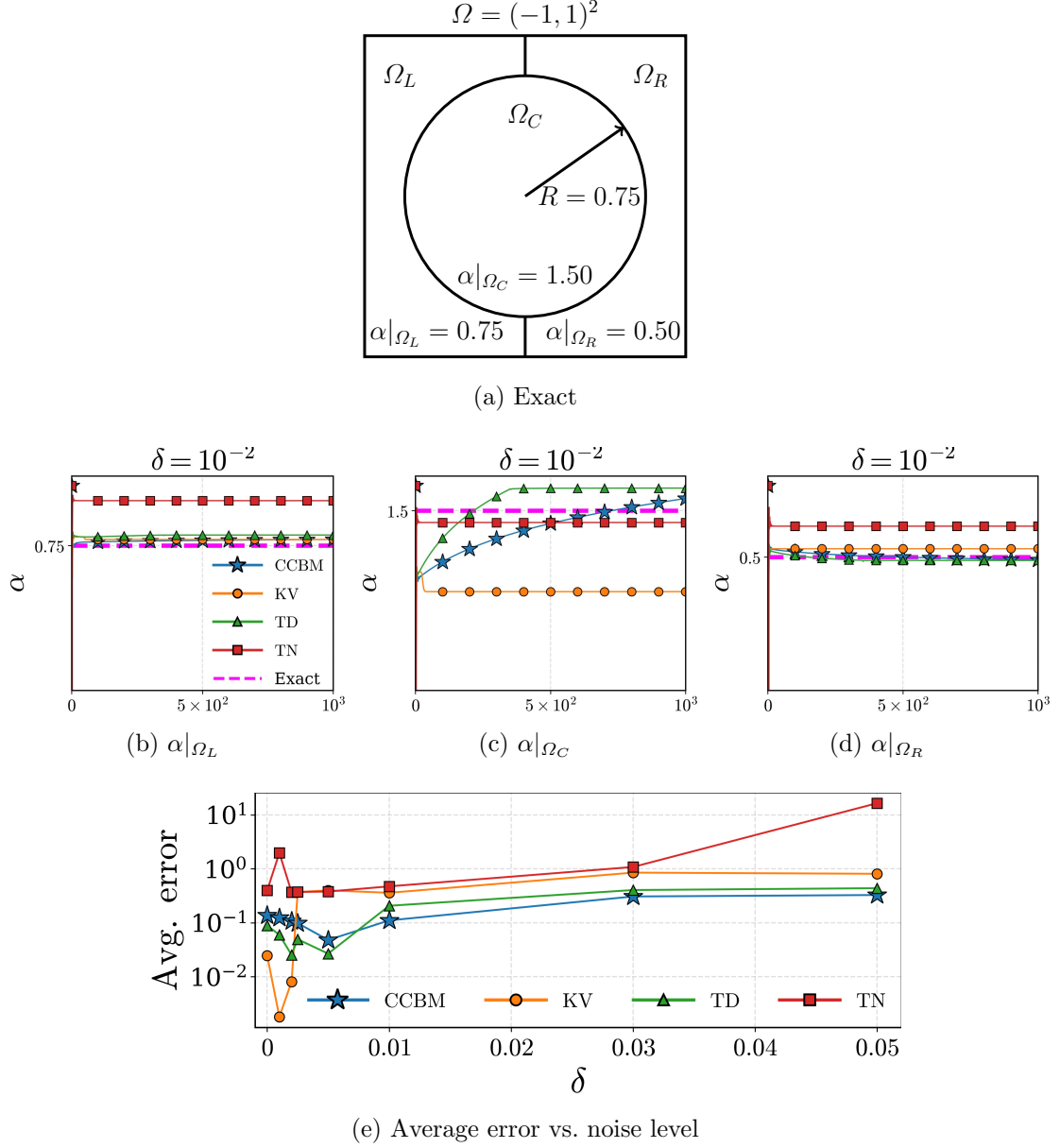


Figure 5: Exact configuration (a) and reconstruction results for $\delta = 10^{-2}$ (b)–(d). Panel (e) shows the average absolute reconstruction error as a function of the noise level δ .

5 Conclusions

We proposed a modified CCBM framework for diffusion coefficient reconstruction from Cauchy data and established theoretical properties of the associated finite-dimensional regularized optimization problem under restrictive assumptions, including continuity and differentiability of the forward map, strict convexity of the cost functional, existence of minimizers, stability with respect to noisy data, and convergence under vanishing noise. Numerical results demonstrate improved reconstruction quality through gradient weighting and estimation of piecewise-constant coefficient values on prescribed partitions via a projection-based extension.

Appendix

This appendix contains supplementary theoretical results and proofs that support the analysis presented in the main text (Appendices A and B), followed by additional numerical experiments (Appendix C). Throughout this appendix, $\|\cdot\|_{s,\Gamma}$ denotes the norm of $H^s(\Gamma)$ for $s \in \mathbb{R}$.

A Theoretical analysis of the forward mapping and inverse problem

This section collects the rigorous proofs and technical results regarding the well-posedness of the forward PDE (Proposition A.1 and Corollary A.2), the differentiability of the parameter-to-state mapping F (Propositions A.3–A.6), and the solvability of the regularized inverse problem (Propositions A.7–A.9).

We begin with coercivity and stability estimates for the variational problem (2.4), proceed to derive higher-order sensitivities of F , and conclude with proofs of existence, strict convexity, and continuous dependence for the objective functional J_ρ .

Proposition A.1. *Let $\Omega \subset \mathbb{R}^d$, $d \in \{1, 2, 3\}$, be a bounded, simply connected domain with Lipschitz boundary $\Gamma := \partial\Omega$. Define the sesquilinear form $\tilde{B} : H^1(\Omega) \times H^1(\Omega) \rightarrow \mathbb{C}$ and the linear functional $\tilde{\ell} : H^1(\Omega) \rightarrow \mathbb{C}$ by*

$$\begin{aligned}\tilde{B}(u, v; \tilde{\alpha}) &= (\tilde{\alpha} \nabla u, \nabla v)_{0,\Omega} + (\tilde{b} \cdot \nabla u, v)_{0,\Omega} + (\tilde{c} u, v)_{0,\Omega} + i \langle u, v \rangle_\Gamma, \\ \tilde{\ell}(v) &= (\tilde{Q}, v)_{0,\Omega} + \langle \tilde{g}, v \rangle_\Gamma + i \langle \tilde{f}, v \rangle_\Gamma.\end{aligned}$$

Assume the data satisfy:

$$(i) \quad \tilde{f}, \tilde{g} \in H^{-1/2}(\Gamma; \mathbb{R}), \quad \tilde{Q} \in L^2(\Omega; \mathbb{R}), \quad \tilde{b} \in W^{1,\infty}(\Omega; \mathbb{R}^d), \quad \tilde{c} \in L^\infty(\Omega; \mathbb{R}), \quad \text{and}$$

$$\tilde{\alpha} \in L_+^\infty(\Omega; \mathbb{R}) := \left\{ \beta \in L^\infty(\Omega) \mid \inf_{x \in \Omega} \beta(x) \geq \beta_0 > 0 \right\};$$

$$(ii) \quad \tilde{b} \cdot n \geq 0 \text{ a.e. on } \Gamma;$$

$$(iii) \quad \tilde{c} - \frac{1}{2} \nabla \cdot \tilde{b} \geq \gamma_0 > 0 \text{ in } \Omega.$$

Then there exists a unique $u \in H^1(\Omega; \mathbb{C})$ such that $\tilde{B}(u, v; \tilde{\alpha}) = \tilde{\ell}(v)$ for all $v \in H^1(\Omega; \mathbb{C})$. Moreover, there exists a constant $C > 0$, depending only on Ω , $\tilde{b}$, and $\tilde{c}$ but independent of $\tilde{\alpha}$, $\tilde{Q}$, $\tilde{f}$, and $\tilde{g}$, such that

$$\|u\|_{1,\Omega} \leq \frac{C}{c_{\tilde{B}}} \left(\|\tilde{Q}\|_{0,\Omega} + \|\tilde{f}\|_{-1/2,\Gamma} + \|\tilde{g}\|_{-1/2,\Gamma} \right),$$

where $c_{\tilde{B}} > 0$ is the coercivity constant, satisfying

$$\Re\{\tilde{B}(v, v; \alpha)\} \geq c_{\tilde{B}} \|v\|_{1,\Omega}^2, \quad \forall v \in H^1(\Omega; \mathbb{C}).$$

Proof. Under assumptions (i)–(iii), a direct computation shows that $\tilde{B}$ is bounded and coercive on $H^1(\Omega; \mathbb{C})$, and that $\tilde{\ell}$ is a bounded linear functional on $H^1(\Omega; \mathbb{C})$. The conclusion then follows immediately from Lax–Milgram theorem on the complex Hilbert space (see, e.g., [17, p. 376]), together with the standard associated stability estimate. $\square$

Corollary A.2 (Well-posedness of Problem 2.4). *Let Assumption 2.2 hold and let $\alpha \in \overline{\mathcal{A}}$, so that $\alpha \geq \underline{\alpha} > 0$ a.e. in Ω . Then the sesquilinear form $B(\cdot, \cdot; \alpha)$ is coercive on $H^1(\Omega)$, i.e., there exists a constant $c_B > 0$, independent of α , such that $\Re\{B(v, v; \alpha)\} \geq c_B \|v\|_{1,\Omega}^2$, for all $v \in H^1(\Omega)$. Consequently, for any $Q \in L^2(\Omega)$, $f \in H^{1/2}(\Gamma)$, and $g \in H^{-1/2}(\Gamma)$, Problem 2.4 admits a unique solution $u \in H^1(\Omega)$, which satisfies the stability estimate*

$$\|u\|_{1,\Omega} \leq \frac{C\lambda}{c_B}, \quad \lambda = \|Q\|_{0,\Omega} + \|f\|_{1/2,\Gamma} + \|g\|_{-1/2,\Gamma}, \quad (\text{A.1})$$

where $C > 0$ depends only on Ω and the coefficients b and c , but is independent of α , Q , f , and g .

Proof. Let $v \in H^1(\Omega)$. Since $\Re\{i(v, v)_{0,\Gamma}\} = 0$, we have

$$\Re\{B(v, v; \alpha)\} = (\alpha \nabla v, \nabla v)_{0,\Omega} + \Re\{(b \cdot \nabla v, v)_{0,\Omega}\} + (cv, v)_{0,\Omega}.$$

Using integration by parts and Assumption 2.2, we obtain

$$\Re\{(b \cdot \nabla v, v)_{0,\Omega}\} = -\frac{1}{2}((\nabla \cdot b)v, v)_{0,\Omega} + \frac{1}{2}(b \cdot nv, v)_{0,\Gamma}.$$

By Assumption 2.2, the boundary term is nonnegative, and by Assumption 2.2,

$$(cv, v)_{0,\Omega} - \frac{1}{2}((\nabla \cdot b)v, v)_{0,\Omega} \geq \gamma_0 \|v\|_{0,\Omega}^2.$$

Moreover, since $\alpha \geq \underline{\alpha}$ a.e. in Ω , $(\alpha \nabla v, \nabla v)_{0,\Omega} \geq \underline{\alpha} \|\nabla v\|_{0,\Omega}^2$. Collecting the estimates yields

$$\Re\{B(v, v; \alpha)\} \geq \underline{\alpha} \|\nabla v\|_{0,\Omega}^2 + \gamma_0 \|v\|_{0,\Omega}^2 \geq \min\{\underline{\alpha}, \gamma_0\} \|v\|_{1,\Omega}^2.$$

Setting $c_B := \min\{\gamma_0, \underline{\alpha}\}$ completes the coercivity proof.

The boundedness of the sesquilinear form $B(\cdot, \cdot; \alpha)$ and of the linear functional, as well as the stability estimate, follow from standard arguments and are therefore omitted. Uniqueness follows from coercivity by standard arguments. $\square$

The following proposition establishes the continuity of the forward map. The result can be verified by a standard argument; hence, the proof is omitted.

Proposition A.3. *The forward map $F: \overline{\mathcal{A}} \subset L^\infty(\Omega) \rightarrow H^1(\Omega)$ is Lipschitz continuous. More precisely, there exists a constant $C > 0$, independent of $\alpha_1, \alpha_2 \in \overline{\mathcal{A}}$, such that*

$$\|F(\alpha_1) - F(\alpha_2)\|_{1,\Omega} \leq \frac{C\lambda}{c_B^2} \|\alpha_1 - \alpha_2\|_{\infty,\Omega}, \quad \text{for all } \alpha_1, \alpha_2 \in \overline{\mathcal{A}}.$$

We next study the differentiability of the forward map $F: \overline{\mathcal{A}} \rightarrow H^1(\Omega)$ defined in (2.2). Let $\alpha \in \mathcal{A}$ and let $\beta \in L^\infty(\Omega)$ be such that $\alpha + \beta \in \overline{\mathcal{A}}$. We define $\delta F := F(\alpha + \beta) - F(\alpha) \in H^1(\Omega)$. By definition of F and subtracting the corresponding variational problems, δF satisfies

$$B(\delta F, v; \alpha + \beta) = -(\beta \nabla u, \nabla v)_{0,\Omega}, \quad \forall v \in H^1(\Omega),$$

where $u = F(\alpha)$. The above variational equation allows us to identify the derivative of F by passing to the limit as $\beta \rightarrow 0$.

Proposition A.4. *Let $\alpha \in \mathcal{A}$. Then the forward map F is Fréchet differentiable at α . For any $\beta \in L^\infty(\Omega)$, the derivative $u' := F'(\alpha)\beta \in H^1(\Omega)$ is the unique solution of*

$$B(u', v; \alpha) = -(\beta \nabla u, \nabla v)_{0,\Omega}, \quad \text{for all } v \in H^1(\Omega),$$

where $u = F(\alpha)$. Moreover, the following estimate holds:

$$\|F'(\alpha)\beta\|_{1,\Omega} \leq \frac{C\lambda}{c_B^2} \|\beta\|_{\infty,\Omega}.$$

Remark A.5. *For $\alpha \in \partial\bar{\mathcal{A}} := \bar{\mathcal{A}} \setminus \mathcal{A}$, denote by $\tilde{\mathcal{A}}$ the largest subset of $L^\infty(\Omega)$ such that for all $\beta \in \tilde{\mathcal{A}}$ and sufficiently small $t > 0$, we have $\alpha + t\beta \in \bar{\mathcal{A}}$. Then the map F is also differentiable at each $\alpha \in \partial\bar{\mathcal{A}}$, and $\|F'(\alpha)\|_{\mathcal{B}(\tilde{\mathcal{A}}, H^1(\Omega))}$ is bounded, where $\mathcal{B}(\tilde{\mathcal{A}}, H^1(\Omega))$ denotes the space of bounded linear operators from $\tilde{\mathcal{A}}$ to $H^1(\Omega)$. That is, one has the operator-norm bound $\|F'(\alpha)\|_{\mathcal{B}(\tilde{\mathcal{A}}, H^1(\Omega))} \leq C\lambda c_B^{-2}$ for all $\alpha \in \bar{\mathcal{A}}$.*

We now turn to the second-order differentiability of the forward map F . We define $\delta^2 F := F'(\alpha + \beta_1)\beta_2 - F'(\alpha)\beta_2 \in H^1(\Omega)$. Using Proposition A.4 and subtracting the corresponding linearized variational problems, we obtain

$$\begin{aligned} B(\delta^2 F, v; \alpha + \beta_1) &= -(\beta_2 \nabla[F(\alpha + \beta_1) - F(\alpha)], \nabla v)_{0,\Omega} \\ &\quad - (\beta_1 \nabla F'(\alpha)\beta_2, \nabla v)_{0,\Omega}, \quad \forall v \in H^1(\Omega). \end{aligned}$$

Passing to the limit as $\|\beta_1\|_{\infty,\Omega} \rightarrow 0$ and using the stability and continuity properties of F and F' (Propositions A.3–A.4), we obtain the second derivative of F at α .

Proposition A.6. *Let $\alpha \in \mathcal{A}$. Then the forward map F is twice Fréchet differentiable at α with respect to admissible directions. For any admissible $\beta_1, \beta_2 \in L^\infty(\Omega)$, the second derivative $u'' := F''(\alpha)[\beta_1, \beta_2] \in H^1(\Omega)$ is the unique solution of*

$$B(u'', v; \alpha) = -(\beta_2 \nabla F'(\alpha)\beta_1, \nabla v)_{0,\Omega} - (\beta_1 \nabla F'(\alpha)\beta_2, \nabla v)_{0,\Omega}, \quad (\text{A.2})$$

where $u = F(\alpha)$. Moreover, there exists a constant $C > 0$, independent of α , such that

$$\|F''(\alpha)[\beta_1, \beta_2]\|_{1,\Omega} \leq \frac{C\lambda}{c_B^3} \|\beta_1\|_{\infty,\Omega} \|\beta_2\|_{\infty,\Omega}. \quad (\text{A.3})$$

In what follows, we establish convexity of J_ρ and uniqueness of the minimizer. Let $\alpha \in \mathcal{K}$ and let $\beta \in \mathcal{K}$ be such that $\alpha + \beta \in \mathcal{K}$. Denote by $u' = F'(\alpha)\beta$ and $u'' = F''(\alpha)[\beta, \beta]$ the first- and second-order variations of the state.

The first Gâteaux derivative of J_ρ in direction β is given by

$$J'_\rho(\alpha)\beta = J'(\alpha)\beta + \rho R'(\alpha)\beta,$$

where

$$J'(\alpha)\beta := w_0(u_i(\alpha), u'_i)_{0,\Omega} + w_1(\nabla u_i(\alpha), \nabla u'_i)_{0,\Omega}, \quad R'(\alpha)\beta := (\alpha, \beta)_{0,\Omega},$$

and $u'_i = \Im\{u'\}$.

The second derivative satisfies

$$\begin{aligned} J''_\rho(\alpha)[\beta, \beta] &= w_0\left(\|u'_i\|_{0,\Omega}^2 + (u_i(\alpha), u''_i)_{0,\Omega}\right) \\ &\quad + w_1\left(\|\nabla u'_i\|_{0,\Omega}^2 + (\nabla u_i(\alpha), \nabla u''_i)_{0,\Omega}\right) + \rho\|\beta\|_{0,\Omega}^2, \end{aligned} \quad (\text{A.4})$$

where $u''_i = \Im\{u''\}$ solves the second-order sensitivity problem (A.2).

Proposition A.7. *Let $\mathcal{K} \subset \mathcal{A}$ be finite-dimensional, closed, and convex (cf. Assumption 2.6). There exists $\rho_0 > 0$, independent of $\alpha \in \mathcal{K}$, such that for all $\rho \geq \rho_0$, the functional J_ρ is strictly convex on $\mathcal{K}$.*

Proof. Let $\alpha, \beta \in \mathcal{K}$ be arbitrary and $\beta \neq 0$ such that $\alpha + \beta \in \mathcal{K}$. Denote by $u' = F'(\alpha)\beta$ and $u'' = F''(\alpha)[\beta, \beta]$ the first- and second-order variations of the state. We prove strict convexity by showing coercivity in the $L^\infty(\Omega)$ -sense of the second derivative $J''_\rho(\alpha)$ in (A.4).

From (A.1) and (A.3), there exist constants $c_0, c_1 > 0$, independent of α , such that

$$\|u_i(\alpha)\|_{0,\Omega}, \|\nabla u_i(\alpha)\|_{0,\Omega} \leq c_0, \quad \|u_i''\|_{0,\Omega}, \|\nabla u_i''\|_{0,\Omega} \leq c_1 \|\beta\|_{\infty,\Omega}^2.$$

On the other hand, since $\mathcal{K}$ is finite-dimensional, there exists $K > 0$ such that $\|\beta\|_{0,\Omega} \geq K\|\beta\|_{\infty,\Omega}$.

Define $J_0 := w_0\|u_i'\|_{0,\Omega}^2 + w_1\|\nabla u_i'\|_{0,\Omega}^2$ and set $c_2 := c_0c_1 > 0$. Then, the second derivative satisfies

$$\begin{aligned} J''_\rho(\alpha)[\beta, \beta] &= J_0 + w_0(u_i(\alpha), u_i'')_{0,\Omega} + w_1(\nabla u_i(\alpha), \nabla u_i'')_{0,\Omega} + \rho\|\beta\|_{0,\Omega}^2 \\ &\geq J_0 - (w_0 + w_1)c_2\|\beta\|_{\infty,\Omega}^2 + \rho\|\beta\|_{0,\Omega}^2 \\ &\geq (\rho K^2 - (w_0 + w_1)c_2)\|\beta\|_{\infty,\Omega}^2. \end{aligned}$$

Now, to ensure a uniform positive lower bound, we first fix any constant $\epsilon > 0$ and define

$$\rho_0 := \frac{(w_0 + w_1)c_2}{K^2} + \epsilon > 0.$$

Then, for all $\rho \geq \rho_0$, we have

$$J''_\rho(\alpha)[\beta, \beta] \geq (\rho K^2 - (w_0 + w_1)c_2)\|\beta\|_{\infty,\Omega}^2 \geq \epsilon K^2\|\beta\|_{\infty,\Omega}^2 > 0,$$

for all $0 \neq \beta \in \mathcal{K}$. Hence, J_ρ is strictly convex on $\mathcal{K}$ uniformly in α , with a uniform lower bound ϵK^2 on the coercivity. $\square$

We now consider the finite-dimensional regularized inverse problem with noisy data, aiming to recover a stable approximation of the diffusion coefficient in $\mathcal{K}$. The first question we address is whether a minimizer actually exists.

Proposition A.8. *Problem 3.2 admits at least one solution $\alpha_\rho^\delta \in \mathcal{K}$.*

Proof. Let g denote the prescribed Neumann boundary data and f the corresponding Dirichlet boundary measurement, both held fixed throughout the following proof. Since $J_\rho(\alpha) \geq 0$ for all $\alpha \in \mathcal{K}$, we can define $j_\rho := \inf_{\alpha \in \mathcal{K}} J_\rho(\alpha) \geq 0$. Let $\{\alpha_n\} := \{\alpha_n\}_{n \in \mathbb{N}} \subset \mathcal{K}$ be a minimizing sequence such that $J_\rho(\alpha_n) \rightarrow j_\rho$ as $n \rightarrow \infty$. By definition of J_ρ and the coercivity of the regularization term, we have

$$J_\rho(\alpha) = J(\alpha) + \rho R(\alpha) = J(\alpha) + \frac{\rho}{2}\|\alpha\|_{0,\Omega}^2 \geq \frac{\rho}{2}\|\alpha\|_{0,\Omega}^2,$$

for any $\alpha \in \mathcal{K}$, since $J(\alpha) \geq 0$. Furthermore, there exists a constant $C > 0$ such that $J_\rho(\alpha_n) \leq C$ for all n . Combining the two inequalities gives $\|\alpha_n\|_{0,\Omega} \leq \sqrt{2C/\rho}$, for all n . This shows that $\{\alpha_n\}$ is bounded in $L^2(\Omega)$.

Since $\mathcal{K} \subset \mathcal{A} \subset L^\infty(\Omega)$ is finite-dimensional (Assumption 2.6), all norms on $\mathcal{K}$ are equivalent and every bounded sequence in $\mathcal{K}$ is relatively compact. Hence, there exists

a subsequence (still denoted by $\{\alpha_n\}$) and some $\alpha_\rho \in \mathcal{K}$ such that $\alpha_n \rightarrow \alpha_\rho$ strongly in $L^2(\Omega)$ and $L^\infty(\Omega)$.

Let $u_n = F(\alpha_n) \in H^1(\Omega)$ and $u_\rho = F(\alpha_\rho) \in H^1(\Omega)$ denote the corresponding state solutions. By the Lipschitz continuity of F (Proposition A.3), we have

$$\|u_n - u_\rho\|_{1,\Omega} \leq \frac{C\lambda}{c_B^2} \|\alpha_n - \alpha_\rho\|_{\infty,\Omega} \rightarrow 0.$$

The functional J_ρ is continuous with respect to $\alpha \in \mathcal{K}$ because $J(\alpha)$ depends continuously on $u_i(\alpha)$ and $\nabla u_i(\alpha)$ (via (2.3)), and $\rho R(\alpha)$ is continuous. Hence, $J_\rho(\alpha_\rho) = \lim_{n \rightarrow \infty} J_\rho(\alpha_n) = j_\rho$.

Thus, $\alpha_\rho \in \mathcal{K}$ attains the minimum of J_ρ , proving existence. $\square$

Having established existence, it is natural to examine the continuous dependence of the minimizer on the boundary data.

Proposition A.9. *Let Assumption 2.6 hold, and let $\rho \geq \rho_0$ as in Proposition A.7. Let $\{f^n\} \subset H^{1/2}(\Gamma)$ satisfy $\|f^n - f^\infty\|_{1/2,\Gamma} \rightarrow 0$ as $n \rightarrow \infty$. For each n , let $\alpha_\rho^n \in \mathcal{K}$ denote the unique minimizer of J_ρ^n corresponding to f^n , and let $u^n := F(\alpha_\rho^n; f^n) \in H^1(\Omega)$ be the corresponding state. Then, as $n \rightarrow \infty$, $\alpha_\rho^n \rightarrow \alpha_\rho^\infty$ strongly in $L^2(\Omega)$, $u^n \rightarrow u^\infty$ strongly in $H^1(\Omega)$, where α_ρ^∞ and u^∞ correspond to f^∞ .*

Proof. Let $\{f^n\} \subset H^{1/2}(\Gamma)$ satisfy

$$\|f^n - f^\infty\|_{1/2,\Gamma} \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

and let $u^n(\alpha) := F(\alpha; f^n)$ denote the unique solution of the weak problem

$$B(u^n(\alpha), v; \alpha) = (Q, v)_{0,\Omega} + \langle g, v \rangle_\Gamma + i \langle f^n, v \rangle_\Gamma, \quad \forall v \in H^1(\Omega),$$

for each $\alpha \in \mathcal{K}$. The corresponding Tikhonov-regularized cost functional is

$$J_\rho^n(\alpha) := \frac{1}{2} \left(w_0 \|u_i^n(\alpha)\|_{0,\Omega}^2 + w_1 \|\nabla u_i^n(\alpha)\|_{0,\Omega}^2 \right) + \frac{\rho}{2} \|\alpha\|_{0,\Omega}^2.$$

By Corollary A.2, the forward map depends continuously on the Dirichlet datum with respect to the $H^{1/2}(\Gamma)$ norm. Hence, uniformly for $\alpha \in \mathcal{K}$,

$$\|u^n(\alpha) - u^\infty(\alpha)\|_{1,\Omega} \leq \frac{C}{c_B} \|f^n - f^\infty\|_{1/2,\Gamma} \rightarrow 0,$$

which implies pointwise convergence of the functionals: $J_\rho^n(\alpha) \rightarrow J_\rho^\infty(\alpha)$. Since $\mathcal{K} \subset L^\infty(\Omega)$ is finite-dimensional, the sequence $\{\alpha_\rho^n\}$ is uniformly bounded and admits a strongly convergent subsequence in $L^2(\Omega)$, say $\alpha_\rho^n \rightarrow \bar{\alpha} \in \mathcal{K}$. By weak lower semicontinuity and since α_ρ^n minimizes J_ρ^n , i.e., $J_\rho^n(\alpha_\rho^n) \leq J_\rho^n(\alpha_\rho^\infty)$, we obtain

$$J_\rho^\infty(\bar{\alpha}) \leq J_\rho^\infty(\alpha_\rho^\infty).$$

Strict convexity of J_ρ^∞ then yields $\bar{\alpha} = \alpha_\rho^\infty$. Hence, the full sequence converges strongly: $\alpha_\rho^n \rightarrow \alpha_\rho^\infty$ in $L^2(\Omega)$.

Finally, since $\{u^n(\alpha_\rho^n)\}$ is bounded in $H^1(\Omega)$, up to a subsequence $u^n(\alpha_\rho^n) \rightharpoonup \bar{u}$ weakly in $H^1(\Omega)$. Passing to the limit in the weak formulation and using $\alpha_\rho^n \rightarrow \alpha_\rho^\infty$ yields

$$B(\bar{u}, v; \alpha_\rho^\infty) = (Q, v)_{0,\Omega} + \langle g, v \rangle_\Gamma + i \langle f^\infty, v \rangle_\Gamma,$$

for all $v \in H^1(\Omega)$, so $\bar{u} = F(\alpha_\rho^\infty; f^\infty)$. Uniqueness of the weak limit then gives

$$u^n(\alpha_\rho^n) \rightharpoonup u^\infty := F(\alpha_\rho^\infty; f^\infty) \quad \text{in } H^1(\Omega).$$

Moreover, by the Lipschitz continuity of F in both α (Proposition A.3) and f (Corollary A.2), one can strengthen the convergence to strong convergence:

$$\|u^n(\alpha_\rho^n) - u^\infty\|_{1,\Omega} \longrightarrow 0 \quad \text{as } n \rightarrow \infty.$$

This completes the proof. $\square$

B Gradient expressions for benchmarks

This section summarizes the state equations, cost functionals, adjoint systems, and corresponding gradient formulas for the four reconstruction methods evaluated in Sections 4.1 and 4.2: Tracking Neumann (TN), Tracking Dirichlet (TD), Kohn–Vogelius (KV), and the Coupled Complex-Boundary Method (CCBM).

To unify the presentation, consider the primary state equation

$$-\nabla \cdot (\alpha \nabla u) + u = Q \quad \text{in } \Omega, \tag{B.1}$$

for a given diffusion coefficient $\alpha \in \overline{\mathcal{A}}$ and source term $Q \in L^2(\Omega)$, equipped with prescribed boundary data $(f, g) \in H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$ forming a Cauchy pair on Γ .

To formalize the action on the boundary traces, we introduce the Neumann-to-Dirichlet map

$$\Lambda_\alpha^{-1} : H^{-1/2}(\Gamma) \rightarrow H^{1/2}(\Gamma), \quad \Lambda_\alpha^{-1} g := u|_\Gamma,$$

which assigns to each Neumann flux g its corresponding Dirichlet trace, where Λ_α denotes the standard Dirichlet-to-Neumann map.

To complete our preparation, we let $u_D, u_N \in H^1(\Omega)$ solve the Dirichlet and Neumann problems, respectively:

$$-\nabla \cdot (\alpha \nabla u_D) + u_D = Q \quad \text{in } \Omega, \quad u_D = f \quad \text{on } \Gamma, \tag{B.2}$$

$$-\nabla \cdot (\alpha \nabla u_N) + u_N = Q \quad \text{in } \Omega, \quad \alpha \partial_n u_N = g \quad \text{on } \Gamma. \tag{B.3}$$

Reconstruction is formulated as the minimization of method-specific cost functionals:

- (i) *Boundary-data tracking:* Let $u_D \in H^1(\Omega)$ solve (B.2); that is, equation (B.1) is equipped with the Dirichlet data f . Then, the Neumann-tracking functional is given by³

$$J_{\text{TN}}(\alpha) := \frac{1}{2} \|\Lambda_\alpha f - g\|_{0,\Gamma}^2.$$

Accordingly, the minimization problem reads

$$\min_{\alpha \in \overline{\mathcal{A}}} J_{\text{TN},\rho}(\alpha), \quad \text{where } J_{\text{TN},\rho}(\alpha) := J_{\text{TN}}(\alpha) + \rho R(\alpha).$$

³While the Neumann-to-Dirichlet operator Λ_α^{-1} maps into $H^{1/2}(\Gamma) \subset L^2(\Gamma)$ unconditionally, the Dirichlet-to-Neumann map Λ_α naturally acts into $H^{-1/2}(\Gamma)$. Thus, defining J_{TN} in the $L^2(\Gamma)$ -norm implicitly assumes $g \in L^2(\Gamma)$ so that $\alpha \partial_n u_D|_\Gamma \in L^2(\Gamma)$.

The Fréchet derivative of J_{TN} at α in the direction $\beta \in L^\infty(\Omega)$ is given by

$$J'_{\text{TN}}(\alpha)[\beta] = \int_{\Omega} \beta \nabla u_{\text{D}} \cdot \nabla p_{\text{D}} \, dx,$$

where the adjoint state $p_{\text{D}} \in H^1(\Omega)$ solves

$$-\nabla \cdot (\alpha \nabla p_{\text{D}}) + p_{\text{D}} = 0 \quad \text{in } \Omega, \quad p_{\text{D}} = \alpha \partial_n u_{\text{D}} - g \quad \text{on } \Gamma.$$

Hence, the L^2 -gradient of J_{TN} is given by

$$\nabla J_{\text{TN}}(\alpha) = \nabla u_{\text{D}} \cdot \nabla p_{\text{D}}.$$

When the Neumann data are prescribed instead (that is, when equation (B.1) is equipped with g , as in (B.3)), we can equivalently track Dirichlet traces via the map Λ_{α}^{-1} :

$$J_{\text{TD}}(\alpha) := \frac{1}{2} \|\Lambda_{\alpha}^{-1} g - f\|_{0,\Gamma}^2,$$

and consider the minimization problem

$$\min_{\alpha \in \bar{\mathcal{A}}} J_{\text{TD},\rho}(\alpha), \quad \text{where } J_{\text{TD},\rho}(\alpha) := J_{\text{TD}}(\alpha) + \rho R(\alpha).$$

The L^2 -gradient of J_{TD} is defined by

$$\nabla J_{\text{TD}}(\alpha) = -\nabla u_{\text{N}} \cdot \nabla p_{\text{N}},$$

where the corresponding adjoint state $p_{\text{N}} \in H^1(\Omega)$ solves

$$-\nabla \cdot (\alpha \nabla p_{\text{N}}) + p_{\text{N}} = 0 \quad \text{in } \Omega, \quad \alpha \partial_n p_{\text{N}} = u_{\text{N}} - f \quad \text{on } \Gamma.$$

(ii) *Kohn–Vogelius method*: The Kohn–Vogelius functional is given by

$$J_{\text{KV}}(\alpha) := \frac{1}{2} \int_{\Omega} (\alpha |\nabla(u_{\text{D}} - u_{\text{N}})|^2 + (u_{\text{D}} - u_{\text{N}})^2) \, dx,$$

and the corresponding optimization problem reads

$$\min_{\alpha \in \bar{\mathcal{A}}} J_{\text{KV},\rho}(\alpha), \quad \text{where } J_{\text{KV},\rho}(\alpha) := J_{\text{KV}}(\alpha) + \rho R(\alpha).$$

Because the functional $J_{\text{KV}}(\alpha)$ is self-adjoint⁴, its L^2 -gradient takes the closed-form expression

$$\nabla J_{\text{KV}}(\alpha) = \frac{1}{2} (|\nabla u_{\text{D}}|^2 - |\nabla u_{\text{N}}|^2).$$

(iii) *Coupled complex-boundary method*: Let $u \in H^1(\Omega; \mathbb{C})$ satisfy (B.1) with the complex Robin condition

$$\alpha \partial_n u + iu = g + if \quad \text{on } \Gamma.$$

⁴No separate adjoint system is required for the Kohn–Vogelius formulation: since u_{D} and u_{N} already solve their respective (self-adjoint) state problems, the sensitivity terms arising from the implicit dependence of $u_{\text{D}}, u_{\text{N}}$ on α cancel identically; see, for example, [2].

The CCBM functional is defined by

$$J_{\text{CCBM}}(\alpha) := \frac{1}{2} \int_{\Omega} \Im\{u\}^2 dx,$$

and the minimization problem reads

$$\min_{\alpha \in \mathcal{A}} J_{\text{CCBM},\rho}(\alpha), \quad \text{where } J_{\text{CCBM},\rho}(\alpha) := J_{\text{CCBM}}(\alpha) + \rho R(\alpha).$$

The L^2 -gradient of J_{CCBM} is given by

$$\nabla J_{\text{CCBM}}(\alpha) = \nabla u_r \cdot \nabla p_i - \nabla u_i \cdot \nabla p_r,$$

where $u = u_r + i u_i$ and $p = p_r + i p_i$ denote the decompositions into real and imaginary parts of the state and adjoint state, respectively. Here, the corresponding adjoint state $p \in H^1(\Omega; \mathbb{C})$ solves

$$-\nabla \cdot (\alpha \nabla p) + p = u_i \quad \text{in } \Omega, \quad \alpha \partial_n p - i p = 0 \quad \text{on } \Gamma.$$

C Additional supporting numerical results

This section provides further numerical evidence supporting the findings presented in the main text. Specifically, the effects of regularization, noise, and domain partitioning are illustrated in Figures C.1–C.2. In the latter figure, we consider a two-subregion configuration on $\Omega = (-1, 1)^2$, partitioned into $\Omega_L = \{x_1 < 0\}$ and $\Omega_R = \{x_1 \geq 0\}$ with diffusion coefficients $\alpha = 0.75$ on Ω_L and $\alpha = 0.50$ on Ω_R , subject to boundary data $g(x_1, x_2) = 1 + 0.5 \sin(\pi x_1) \sin(\pi x_2)$.

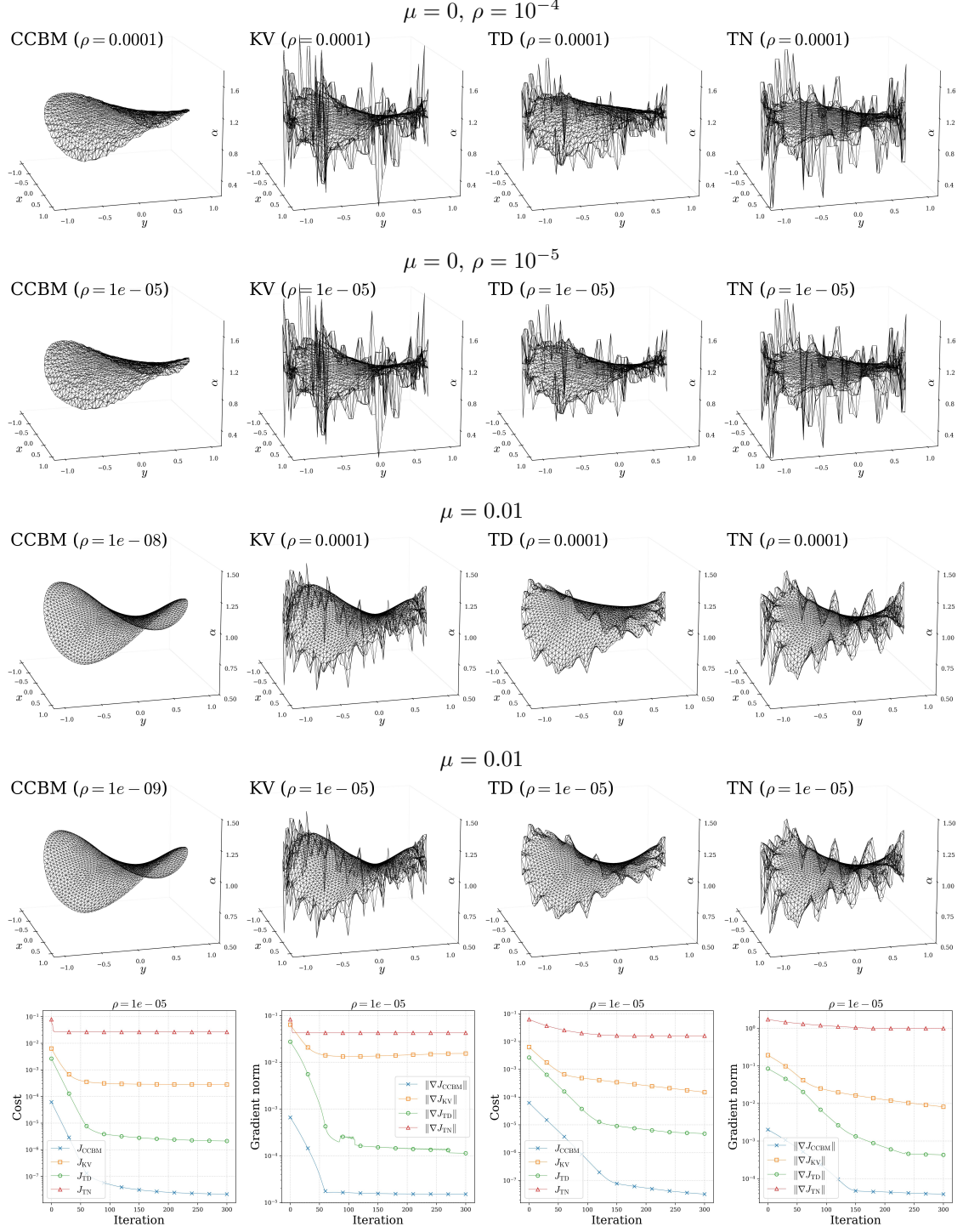


Figure C.1: Effect of the Tikhonov parameter ρ on the reconstruction ($\delta = 0.0003$) and optimization history. Rows 1–2: $\mu = 0$ (no H^1 smoothing) with $\rho = 10^{-4}$ and $\rho = 10^{-5}$, respectively. Rows 3–4: $\mu = 0.01$ (with H^1 smoothing) at the corresponding ρ values. Bottom row (left to right): cost history and gradient norm history without smoothing, followed by cost history and gradient norm history with smoothing.

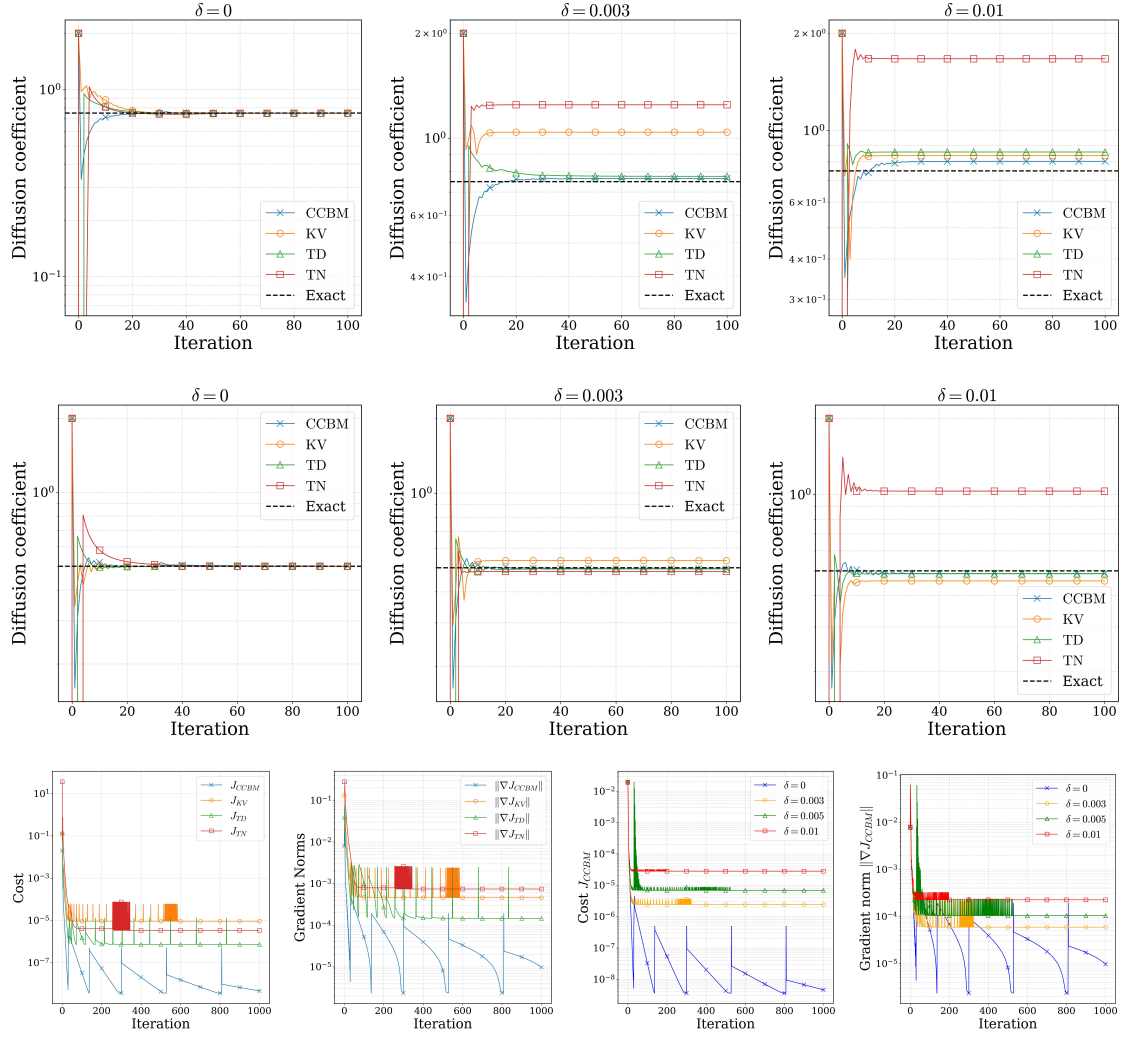


Figure C.2: Two-subregion case. Top row: reconstructed diffusion coefficient in Ω_L for noise levels $\delta = 0, 0.003$, and 0.01 (left to right). Middle row: corresponding reconstructions in Ω_R . Bottom row (left to right): cost functional and gradient norm histories for all methods with exact data ($\delta = 0$), followed by CCBM cost functional and gradient norm histories across the noise levels $\delta = 0, 0.003, 0.01$. Dashed lines indicate the exact values.

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